JOURNAL OF COMPUTATIONAL PHYSICS 1, 51-67 (1966)

# Stability of Difference Approximations to Certain Partial Differential Equations of Fluid Dynamics

DAVID WILLIAMSON

*National Center for Atmospheric Research, Boulder, Colorado*

## ABSTRACT

First- and second-order difference approximations to certain partial differential equations of fluid dynamics are investigated for the von Neumann necessary condition for stability. The nonsteady equations examined consist of advection, diffusion, and inertial terms. Although the general equation considered represents no particular atmospheric process, it does have features found in many meteorological problems. Five approximations to the diffusion equation are studied, three of which are shown to be stable. Two of the three approximations to the advection-diffusion equation investigated are found to be stable. Twelve approximations to the advection-inertial equation are examined; eight are found to be stable and two are found to be slightly unstable. One two-step scheme, each step of which is individually stable, is shown to be unstable. For the advection-diffusion-inertial equation, 2 two-step schemes are formed from the preceding stable schemes. The analysis shows that such combinations are not necessarily stable.

## INTRODUCTION

The increasing interest in the use of second-order difference equations for numerical problems in fluid dynamics has made it desirable to study the stability properties of such approximations. Also of interest is the stability of more complicated difference schemes of both first- and second-order approximating nonsteady partial differential equations containing several terms in addition to the time derivative. Such schemes arise from the desire to solve systems of partial differential equations appearing, for example, in problems of the general circulation of atmospheres and oceans. These systems of equations often include, among others, advection, diffusion, and inertial terms. In this paper a nonsteady partial differential equation containing these terms is considered, which, while not approximating any particular atmospheric process, does have characteristics similar to equations of meteorology. Difference approximations to various simplifications of this equation are first considered, followed by the effect on the stability of combining these simplifications for the more general equation.

Richtmyer [8] has summarized the stability of several difference approximations to the advection equation. This summary includes the unstable forward difference; the stable diffusing and upstream differencing, all first-order-in-time; and the stable second-order-in-time leap frog and Lax-Wendroff schemes. Richtmyer [7] has also given the stability properties of various first- and second-order approximations to the diffusion equation. The following report deals with the von Neumann necessary condition for stability of some additional first- and second-order approximations to the diffusion equation. Several difference equations are formed by combining stable schemes for an advection term and for a diffusion term, and by combining stable schemes for an advection term and for an inertial term in nonsteady equations. The stability of these equations is examined. Finally, schemes found to be stable are combined into schemes for an advection-diffusion-inertial equation and their stability is investigated.

The equations examined are all derived from the following equations:

$$\frac{\partial u}{\partial t} + \hat{A} \frac{\partial u}{\partial x} = \hat{K} \frac{\partial^2 u}{\partial x^2} + \hat{f} v \tag{1}$$

$$\frac{\partial v}{\partial t} + \hat{A} \frac{\partial v}{\partial x} = \hat{K} \frac{\partial^2 v}{\partial x^2} - \hat{f} u, \tag{2}$$

where $u$ and $v$ are eastward and northward velocity components, respectively; $\hat{A}$ is the advection speed, $\hat{K} \geq 0$ is the diffusion coefficient, and $\hat{f}$ is the Coriolis parameter given by $\hat{f} = 2\Omega \sin \varphi$, with $\Omega$ denoting the earth's angular velocity, and $\varphi$, latitude. The parameters $\hat{A}$, $\hat{K}$, and $\hat{f}$ are all assumed to be constant. Combining (1) and (2) using the complex notation $w = u + iv$ results in the general equation

$$\frac{\partial w}{\partial t} + \hat{A} \frac{\partial w}{\partial x} - \hat{K} \frac{\partial^2 w}{\partial x^2} + i\hat{f} w = 0. \tag{3}$$

The second term of (3) is referred to as the advection term, the third as the diffusion term, and the fourth as the inertial term. Although Eq. (3) does not represent any particular atmospheric process, it is similar in many ways to equations of meteorology, and therefore the study of stability of the difference equations is useful.

## I. Stability

It can be shown that the solutions of (3) remain bounded with time. Therefore, for a finite-difference approximation of (3) to be called numerically stable, its solution should consist of Fourier components whose amplitudes do not grow unboundedly with time. We shall call a scheme stable if all its Fourier components remain bounded with time, and hence unstable if at least one component is not bounded.

In this paper two types of approximation commonly used in meteorology are considered, one with two time levels and one with three. In general, one method is not necessarily better than the other. Stability calculations for one equation of each type are given below to illustrate the method.

For the two-time-level approximation we consider (3) with $\hat{A} = \hat{f} = 0$:

$$\frac{\partial u}{\partial t} = \hat{K} \frac{\partial^2 u}{\partial x^2}. \tag{1.1}$$

One scheme approximating (1.1) is

$$U_j^{n+1} = \frac{1}{2} (U_{j+1}^n + U_{j-1}^n) + \hat{K} \frac{\Delta t}{(\Delta x)^2} (U_{j+1}^n - 2U_j^n + U_{j-1}^n). \tag{1.2}$$

For the difference schemes examined in this report a rectangular net in the $x$-$t$ plane is used with spacings $\Delta x$ and $\Delta t$. Upper case letters are used for the calculated values of the dependent variable, and $U_j^n$ denotes the value of $U(j\,\Delta x, n\,\Delta t)$ for integer and half-integer values of $j$ and $n$. To simplify notation, the following abbreviations are introduced:

$$\begin{aligned} \overline{U_j^n} &\equiv \tfrac{1}{2} (U_{j+1}^n + U_{j-1}^n) & \delta_2 U_j^n &\equiv \tfrac{1}{2} (U_{j+1}^n - U_{j-1}^n) \\ \delta_1 U_j^n &\equiv U_{j+1/2}^n - U_{j-1/2}^n & \delta_2^m U_j^n &\equiv \delta_2(\delta_2^{m-1} U_j^n). \\ \delta_1^m U_j^n &\equiv \delta_1(\delta_1^{m-1} U_j^n) & & \end{aligned} \tag{1.3}$$

These abbreviations are also used with $U$ replaced by $W$. The following nondimensional variables are also introduced:

$$A \equiv \hat{A} \frac{\Delta t}{\Delta x}$$
$$K \equiv \hat{K} \frac{\Delta t}{(\Delta x)^2} \tag{1.4}$$
$$f \equiv \hat{f} \Delta t.$$

It should be remembered that the circumflex indicates a dimensional variable, and the absence of a circumflex, a nondimensional variable.

Using abbreviations (1.3) and (1.4), Eq. (1.2) can be written as

$$U_j^{n+1} = \overline{U_j^n} + K\delta_1^2 U_j^n. \tag{1.5}$$

Substituting a Fourier term $U(x, n\,\Delta t) = U^n \exp(ikx)$ into (1.5) results in

$$U^{n+1} = GU^n = G^{n+1}U^0,$$

where $G = \cos\alpha + 2K(\cos\alpha - 1)$ and $\alpha = k\,\Delta x$. $G$ is called the amplification factor. In order for the amplitudes of the Fourier components not to grow unboundedly, $G$ must not exceed unity in absolute value. This is essentially the von Neumann necessary condition for a damping system.[1] Hence, the stability condition for (1.2) is

$$|\cos\alpha + 2K(\cos\alpha - 1)| \leq 1.$$

For $\alpha = \pi$ and $K > 0$, $|G| > 1$. Therefore, there are some Fourier components which grow and scheme (1.2) is unstable.

We now consider a three time-level difference approximation to (3) for the case $\hat{K} = 0$.

$$\frac{\partial w}{\partial t} = -\hat{A}\frac{\partial w}{\partial x} - i\hat{f}w. \tag{1.6}$$

One scheme approximating (1.6) is

$$W_j^{n+1} = W_j^{n-1} - 2A\delta_2 W_j^n - 2if\,W_j^n. \tag{1.7}$$

[1] For problems containing a mechanism causing a growth of the true solution, a more generous condition that the amplification factor should not exceed $1 + O(\Delta t)$ should be used (e. g., see [7]).

Again substituting a Fourier term $W(x, n\,\Delta t) = W^n \exp(ikx)$ into (1.7) results in

$$W^{n+1} = GW^n + HW^{n-1},$$

where $G = -2i(A \sin \alpha + f)$ and $H = 1$. Since the identity $W^n = (1)$ $W^n + (0)\ W^{n-1}$ holds, we can write $\mathbf{W}^{n+1} = \mathbf{M}\mathbf{W}^n$, where

$$\mathbf{W}^n = \begin{pmatrix} W^n \\ W^{n-1} \end{pmatrix}$$

and

$$\mathbf{M} = \begin{pmatrix} G & H \\ 1 & 0 \end{pmatrix}.$$

$\mathbf{M}$ is called the amplification matrix of the vector $\mathbf{W}^n$. In this case, the von Neumann necessary condition for stability is that the eigenvalues of $\mathbf{M}$ do not exceed unity in magnitude. If we denote by $\lambda_{1,2}$ the two eigenvalues of $\mathbf{M}$, then

$$\lambda_{1,2} = -iB \pm (-B^2 + 1)^{1/2},$$

where $B = A \sin \alpha + f$. If $|B| > 1$, the radical is imaginary and one of the eigenvalues exceeds unity in absolute value. However, if $|B| \leq 1$, the radical is real and $|\lambda_{1,2}| = 1$. Thus, scheme (1.7) is stable if

$$|\hat{A}| \frac{\Delta t}{\Delta x} + \hat{f}\,\Delta t \leq 1 \quad \text{or} \quad |A| + f \leq 1.$$

For a few difference equations stability conditions such as the above are not easily found analytically. In these cases, the eigenvalues of the amplification matrix are calculated for various values of $A$, $K$, and $f$, and a graph showing the domain of stability is constructed.

## II. Truncation Error

Truncation error has been defined several ways by different authors. Henrici [1], for example, defines it to be the difference between the exact solution of the differential equation and the exact solution of the difference equation. In this paper, we will use the definition of Richtmyer [7] dealing directly with the difference and differential equations themselves rather than with their solutions. The truncation error of (1.7) approximating (1.6) is calculated as an example.

The truncation error, $e$, of (1.7) in calculating the value of $W_j^{n+1}$ is defined to be the difference

$$e = \frac{W_j^{n+1} - W_j^{n-1}}{2\Delta t} + \hat{A}\frac{W_{j+1}^n - W_{j-1}^n}{2\Delta x} + if W_j^n - \left(\frac{\partial w}{\partial t} + \hat{A}\frac{\partial w}{\partial x} + ifw\right)_j^n. \qquad (2.1)$$

Assuming the value of $W_j^n$ is exact and substituting Taylor's series expansions about $w_j^n$ for $W_j^{n+1}$, $W_j^{n-1}$, $W_{j+1}^n$, and $W_{j-1}^n$ into (2.1) results in

$$e = \left(\frac{\partial w}{\partial t}\right)_j^n + \frac{1}{12}(\Delta t)^2\left[\left(\frac{\partial^3 w}{\partial t^3}\right)_j^{n+\theta_1} + \left(\frac{\partial^3 w}{\partial t^3}\right)_j^{n-\theta_3}\right]$$
$$+ \hat{A}\left\{\left(\frac{\partial w}{\partial x}\right)_j^n + \frac{1}{12}(\Delta x)^2\left[\left(\frac{\partial^3 w}{\partial x^3}\right)_{j+\theta_2}^n + \left(\frac{\partial^3 w}{\partial x^3}\right)_{j-\theta_4}^n\right]\right\}$$
$$+ if w_j^n - \left[\frac{\partial w}{\partial t} + \hat{A}\frac{\partial w}{\partial x} + ifw\right]_j^n;$$
$$0 < \theta_i < 1, \qquad i = 1, ..., 4.$$

Therefore, the truncation error is written as

$$e = O((\Delta t)^2) + O((\Delta x)^2).$$

## III. Difference Equations

In general there are many possible difference equations approximating any one partial differential equation. In this paper only schemes of the type used in meteorological problems are considered.

### A. *Diffusion Equation*

Several difference equations approximating the diffusion equation

$$\frac{\partial u}{\partial t} = \hat{K}\frac{\partial^2 u}{\partial x^2}$$

along with truncation errors, amplification factors or eigenvalues of the amplification matrices, and the stability conditions are summarized below. Since the main purpose of this investigation is to find stable difference schemes, when an instability is found in a scheme for certain Fourier components further investigation is not carried out. Therefore, statements such as "unstable for short wavelengths" imply nothing about the stability of longer wavelengths.

$$U_j^{n+1} = \overline{U_j^n} + K\delta_1^2 U_j^n \tag{A.1}$$
$$e = O(\Delta t) + O[(\Delta x)^2] + O\left[\frac{(\Delta x)^2}{\Delta t}\right]$$
$$G = \cos\alpha + 2K(\cos\alpha - 1).$$

This scheme is unstable for short wavelengths with $\cos\alpha < -(1 - 2K)/(1 + 2K)$.

$$U_j^{n+1} = \overline{U_j^n} + K\delta_1^2 U_j^{n-1} \tag{A.2}$$
$$e = O(\Delta t) + O[(\Delta x)^2] + O\left[\frac{(\Delta x)^2}{\Delta t}\right]$$
$$2\lambda_{1,2} = \cos\alpha \pm [\cos^2\alpha + 8K(\cos\alpha - 1)]^{1/2}.$$

Scheme (A.2) is stable for $K \leq 1/10$.

$$U_j^{n+1} = U_j^n + K\delta_1^2 U_j^n + \tfrac{1}{2} K^2\delta_1^4 U_j^n \tag{A.3}$$
$$e = O[(\Delta t)^2] + O[(\Delta x)^2]$$
$$G = 1 + 2K(\cos\alpha - 1) - 2K^2[\sin^2\alpha + 2(\cos\alpha - 1)].$$

The above scheme is stable for $K \leq \frac{1}{2}$. Equation (A.3), of the type referred to as Lax-Wendroff, is stable for the same values of $K$ as the first-order forward difference scheme as given in [7].

$$U_j^{n+1} = \overline{U_j^n} + K\delta_1^2 U_j^n \tag{A.4}$$
$$U_j^{n+2} = U_j^n + 2K\delta_1^2 U_j^{n+1}$$
$$e = O[(\Delta t)^2] + O[(\Delta x)^2]$$
$$G = 1 + 2K(\cos 2\alpha - 2\cos\alpha + 1) + 4K^2(\cos 2\alpha - 4\cos\alpha + 3).$$

The two-step scheme (A.4) is unstable for short wavelengths. This two-step scheme and those to follow are considered as one difference equation for the stability analysis by eliminating the values of $U$ at time $(n+1)\Delta t$ from the two equations. In actual practice the two equations are used alternately, but the truncation error and stability apply only at every second time step.

$$U_j^{n+1} = U_j^n + K\delta_1^2 U_j^n \tag{A.5}$$
$$U_j^{n+2} = U_j^n + 2K\delta_1^2 U_j^{n+1}.$$

Two-step scheme (A.5) is a combination of the stable forward difference and the unstable leap frog schemes given in [7]. When the first step is substituted into the second, the resulting equation is seen to be the Lax-Wendroff type equation (A.3) with time step $2\Delta t$ and space increment $\Delta x$.

B. *Advection-Diffusion Equation*

The properties of several difference equations approximating an advection-diffusion equation

$$\frac{\partial u}{\partial t} + \hat{A}\frac{\partial u}{\partial x} = \hat{K}\frac{\partial^2 u}{\partial x^2}$$

are given below.

$$\begin{aligned} U_j^{n+1} &= U_j^n - A\delta_2 U_j^n + \tfrac{1}{2} A^2\delta_1^2 U_j^n - AK\delta_1^2(\delta_2 U_j^n) \\ &\quad + K\delta_1^2 U_j^n + \tfrac{1}{2} K^2\delta_1^4 U_j^n \\ e &= O[(\Delta t)^2] + O[(\Delta x)^2] \\ G &= 1 + (A^2 + 2K - 4K^2)(\cos\alpha - 1) - 2K^2\sin^2\alpha \\ &\quad - i[2AK(\cos\alpha - 1) + A]\sin\alpha. \end{aligned} \tag{B.1}$$

No stability criterion can easily be found analytically for the Lax-Wendroff type scheme (B. 1). The domain of stability, as calculated on a computer, is shown in Fig. 1.

$$\begin{aligned} U_j^{n+1} &= \overline{U_j^n} - A\delta_2 U_j^n + K\delta_1^2 U_j^n \\ U_j^{n+2} &= U_j^n - 2A\delta_2 U_j^{n+1} + 2K\delta_1^2 U_j^{n+1} \\ e &= O[(\Delta t)^2] + O[(\Delta x)^2] \\ G &= 1 + A^2(\cos 2\alpha - 1) + 2K(\cos 2\alpha - 2\cos\alpha + 1) \\ &\quad + 4K^2(\cos 2\alpha - 4\cos\alpha + 3) \\ &\quad - i[A\sin 2\alpha + 2AK(2\sin 2\alpha - 4\sin\alpha)] \end{aligned} \tag{B.2}$$

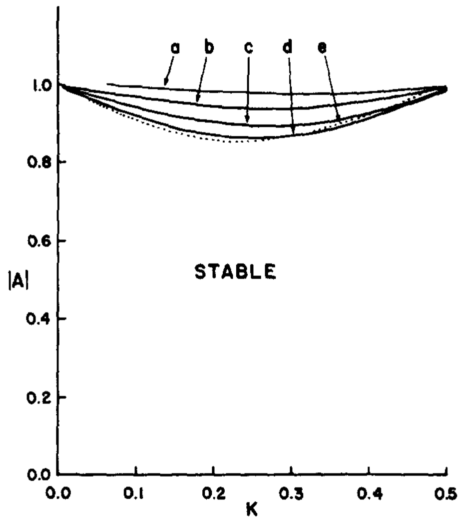


FIG. 1. Domain of stability of scheme (B.1). Stable for $|A| \leq 1$, $K \leq \frac{1}{2}$; and $\alpha \leq \frac{2}{3}\pi$. Stable below lines for: $a$, $\alpha = \frac{25}{36}\pi$; $b$, $\alpha = \frac{29}{36}\pi$; $c$, $\alpha = \frac{5}{6}\pi$; $d$, $\alpha = \frac{11}{12}\pi$; $e$, $\alpha = \pi$.

The above scheme is unstable for short wavelengths when $K > 0$. The stable case, $K = 0$, has been studied in [8].

$$\begin{aligned} U_j^{n+1} &= \overline{U_j^n} - A\delta_2 U_j^n + K\delta_1^2 U_j^n \qquad \text{(B.3)} \\ U_j^{n+2} &= U_j^n - 2A\delta_2 U_j^{n+1} + 2K\delta_1^2 U_j^n \\ e &= O(\Delta t) + O[(\Delta x)^2] \\ G &= 1 - 2A^2 \sin^2 \alpha + 4K(\cos\alpha - 1) \\ &\quad - i[A \sin 2\alpha + 2AK(\sin 2\alpha - 2\sin\alpha)] \end{aligned}$$

Equation (B. 3) is stable for $A^4 + (4K^2 + 4K - 1)A^2 \leq 2K - 4K^2$ and $2K \leq \frac{1}{2}$. Figure 2 shows a graph of the stable region. When the first equation of (B.3) is substituted into the second, the resulting equation is a one-step scheme with time step $2\Delta t$, and the differences in the advection term taken over a grid interval of $2\Delta x$ while those in the diffusion term are taken over a grid interval of $\Delta x$. Therefore, $A = \hat{A}(2\Delta t/2\Delta x)$ and $2K = \hat{K}(2\Delta t/\Delta x)$ are chosen as coordinates of Fig. 2 to make it comparable with the previous results. A similar choice of coordinates is made in other schemes.

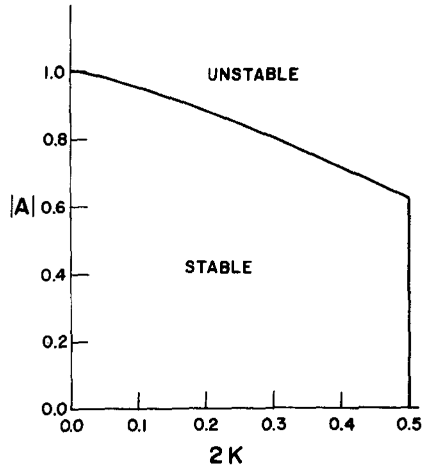


FIG. 2. Domain of stability of scheme (B. 3).

### C. *Advection-Inertial Equation*

Difference schemes approximating an advection-inertial equation

$$\frac{\partial w}{\partial t} + \hat{A}\frac{\partial w}{\partial x} = -i\hat{f}w$$

and their properties are listed below.

$$W_j^{n+1} = \overline{W_j^n} - A\delta_2 W_j^n - if\, W_j^n \tag{C.1}$$

$$e = O(\Delta t) + O[(\Delta x)^2] + O\left[\frac{(\Delta x)^2}{\Delta t}\right]$$

$$G = \cos\alpha - i(A\sin\alpha + f).$$

Scheme (C.1) is unstable for very long and short wavelengths. For $\alpha = 0$ or $\pi$, $|G| = (1+f^2)^{1/2}$; and for $\alpha = \pi/2$, $|G| \leq |A| + f$.

$$W_j^{n+1} = \overline{W_j^n} - A\delta_2 W_j^n - if\, W_j^{n+1} \tag{C.2}$$

$$e = O(\Delta t) + O[(\Delta x)^2] + O\left[\frac{(\Delta x)^2}{\Delta t}\right]$$

$$G(1 + if) = \cos\alpha - iA\sin\alpha.$$

This equation is stable for $|A|^2 \leq 1 + f^2$. As can be seen in the expression for $G$, advancing the time level of the inertial term produces a slight damping of the solution.

$$W_j^{n+1} = \overline{W_j^n} - A\delta_2 W_j^n - \tfrac{1}{2} if (W_j^{n+1} + W_j^n) \qquad \text{(C.3)}$$
$$e = O(\Delta t) + O[(\Delta x)^2] + O\left[\frac{(\Delta x)^2}{\Delta t}\right]$$
$$G = (1 + \tfrac{1}{2} if) = \cos\alpha - i(A \sin\alpha + \tfrac{1}{2} f).$$

Equation (C.3) is stable for $4f \mid A \mid + 3A^2 \leq 3$.

$$W_j^{n+1} = W_j^n - A\delta_2 W_j^n - if W_j^{n+1} \qquad \text{(C.4)}$$
$$e = O(\Delta t) + O[(\Delta x)^2]$$
$$G(1 + if) = 1 - iA \sin\alpha.$$

The above equation is stable for $\mid A \mid \leq f$. For $f = 0$, the equation is unstable, which is well known. For $A = 0$, the scheme is stable as discussed by Kurihara [4]. As was seen in (C.2), evaluating the inertial term at time level $n + 1$ produces a damping which counteracts the instability of the advection term. In this case the stability condition places no restriction on the time increment; however, it does place a lower limit on the space increment. When the advection speed $\hat{A} = 30$ msec$^{-1}$ and $\hat{f} = 10^{-4}$ sec$^{-1}$, $\Delta x$ must exceed 300 km. This lower limit increases as $\hat{A}$ increases or as $\hat{f}$ decreases.

$$W_j^{n+1} = W_j^{n-1} - 2A\delta_2 W_j^n - 2if W_j^n \qquad \text{(C.5)}$$
$$e = O[(\Delta t)^2] + O[(\Delta x)^2]$$
$$\lambda_{1,2} = - i(A \sin\alpha + f) \pm [- (A \sin\alpha + f)^2 + 1]^{1/2}$$

Scheme (C.5) is stable for $\mid A \mid + f \leq 1$. This scheme is also discussed by Kurihara [4].

$$W_j^{n+1} = W_j^{n-1} - 2A\delta_2 W_j^n - 2if W_j^{n+1} \qquad \text{(C.6)}$$
$$e = O(\Delta t) + O[(\Delta x)^2]$$
$$(1 + 4f^2)\lambda_{1,2}^2 + 2iA \sin\alpha(1 - 2if)\lambda_{1,2} - (1 - 2if) = 0.$$

Equation (C.6) is stable for $\mid A \mid < f$. This is the same condition as (C.4).

$$W_j^{n+1} = W_j^{n-1} - 2A\delta_2 W_j^n - if (W_j^{n+1} + W_j^{n-1}) \qquad \text{(C.7)}$$
$$e = O[(\Delta t)^2] + O[(\Delta x)^2]$$
$$\lambda_{1,2}(1 + if) = - iA \sin\alpha \pm [- A^2 \sin^2\alpha + (1 + f^2)]^{1/2}.$$

The above scheme is stable for $A^2 \leq 1 + f^2$.

$$W_j^{n+1} = W_j^n - A\delta_2 W_j^n + \tfrac{1}{2} A^2 \delta_1^2 W_j^n + ifA\delta_2 W_j^n \tag{C.8}$$
$$- ifW_j^n - \tfrac{1}{2} f^2 W_j^n$$
$$e = O[(\Delta t)^2] + O[(\Delta x)^2]$$
$$G = 1 + A^2(\cos\alpha - 1) - Af\sin\alpha - \tfrac{1}{2} f^2 - i(A\sin\alpha + f).$$

Equation (C.8) is of the Lax-Wendroff type. For $f = 0$ it is stable when $|A| < 1$ as given in [8]. For $A = 0$, the equation is slightly unstable with $|G|^2 = 1 + \frac{1}{4} f^4$; however, for $\hat{f} = 10^{-4}$ sec$^{-1}$ and $\Delta t = 10$ minutes, ten thousand iterations could be performed before the amplitude even doubled.

$$W_j^{n+1} = \overline{W_j^n} - A\delta_2 W_j^n - ifW_j^n \tag{C.9}$$
$$W_j^{n+2} = W_j^n - 2A\delta_2 W_j^{n+1} - 2ifW_j^{n+1}$$
$$e = O[(\Delta t)^2] + O[(\Delta x)^2]$$
$$G = 1 + A^2(\cos 2\alpha - 1) - 4Af\sin\alpha - 2f^2$$
$$- i(A\sin 2\alpha + 2f\cos\alpha).$$

This scheme is slightly unstable for both very long and short wavelengths. For $\alpha = 0$ or $\pi$, $|G|^2 = 1 + 4f^4$. Houghton, Kasahara, and Washington [2] have calculated the value of $(1 + 4f^4)^{N/2}$ for various values of $N$ and $f$. In particular, for $N = 10000$, $\hat{f} = 10^{-4}$ sec$^{-1}$, and $\Delta t = 10$ minutes, $(1 + 4f^4)^{N/2}$ is 1.2958847. For most problems this amplification is insignificant. The case $f = 0$ is stable as given in [8]. This first step of (C.9) is the unstable scheme (C.1), and the second is the stable scheme (C.5). As is seen above, the result of combining the two is just slightly unstable.

$$W_j^{n+1} = \overline{W_j^n} - A\delta_2 W_j^n - ifW_j^{n+1} \tag{C.10}$$
$$W_j^{n+2} = W_j^n - 2A\delta_2 W_j^{n+1} - 2ifW_j^{n+1}$$
$$e = O(\Delta t) + O[(\Delta x)^2]$$
$$G(1 + if) = (1 + if) + A^2(\cos 2\alpha - 1) - 2fA\sin\alpha$$
$$- i(A\sin 2\alpha + 2f\cos\alpha)$$

For $\alpha = 0$ scheme (C. 10) is neutral; however, for $\alpha = \pi$, $|G|^2$ is of the order $(1 + 8f^2)$. The values of $(1 + 4f^2)^{N/2}$ are also calculated in [2] and significant amplification is shown after 1000 time steps. The two steps of (C.10) are the stable schemes (C.2) and (C.5). Surprisingly, the resulting two-step scheme is unstable.

$$W_j^{n+1} = \overline{W_j^n} - A\delta_2 W_j^n - ifW_j^{n+1} \qquad \text{(C.11)}$$

$$W_j^{n+2} = W_j^n - 2A\delta_2 W_j^{n+1} - 2ifW_j^{n+2}$$

$$e = O(\Delta t) + O[(\Delta x)^2]$$

$$G(1 + 2if)(1 + if) = (1 + if) + A^2(\cos 2\alpha - 1) - iA \sin \alpha$$

Figure 3 shows the stable values of $A$ and $f$. The one-step equation obtained by substituting the first equation of (C.11) into the second has time step $2\Delta t$ and space increment $2\Delta x$. Therefore, the abscissa of Fig. 3

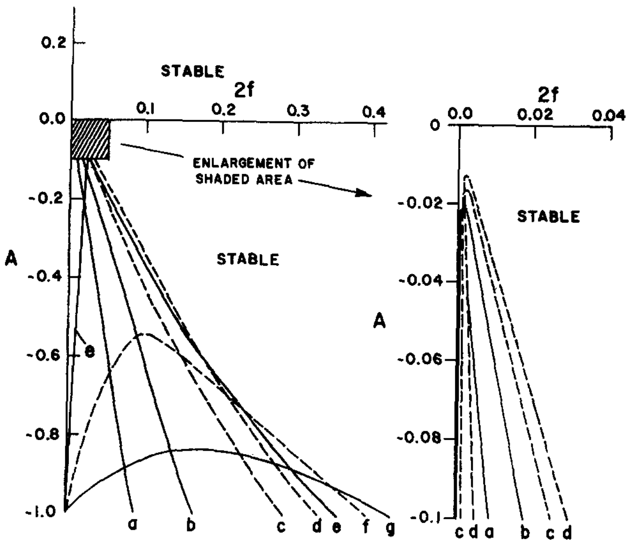


FIG. 3. Domain of stability of scheme (C.11). Stable for $|A| \leq 1, f \leq \frac{1}{2}$, $2\alpha = 0$. Stable above lines for: $a$, $2\alpha = \frac{1}{18}\pi$; $b$, $2\alpha = \frac{1}{9}\pi$; $c$, $2\alpha = \frac{1}{6}\pi$; $d$, $2\alpha = \frac{2}{9}\pi$; $e$, $2\alpha = \frac{5}{18}\pi$; $f$, $2\alpha = \frac{1}{3}\pi$; $g$, $2\alpha = \frac{4}{9}\pi$. Unstable area decrease as $2\alpha \to \pi$.

is chosen to be $2f$ to make it comparable with the results of the preceeding one-step schemes. Similar coordinates are used in Fig. 4. Scheme (C.11) is a combination of the stable schemes (C.2) and (C.6), and, as is seen in Fig. 3, is stable for certain values of the parameters.

$$W_j^{n+1} = \overline{W_j^n} - A\delta_2 W_j^n - \tfrac{1}{2}\, if\,(W_j^{n+1} + \overline{W_j^n}) \qquad \text{(C.12)}$$
$$W_j^{n+2} = W_j^n - 2A\delta_2 W_j^{n+1} - if\,(W_j^{n+2} + W_j^n)$$
$$e = O(\Delta t) + O[(\Delta x)^2]$$
$$G(1 + if)\,(1 + \tfrac{1}{2}\, if) = (1 - if)\,(1 + \tfrac{1}{2}\, if) + A^2(\cos 2\alpha - 1) - iA(1 - \tfrac{1}{2}\, if)\sin 2\alpha$$

Figure 4 shows the domain of stability of scheme (C.12).

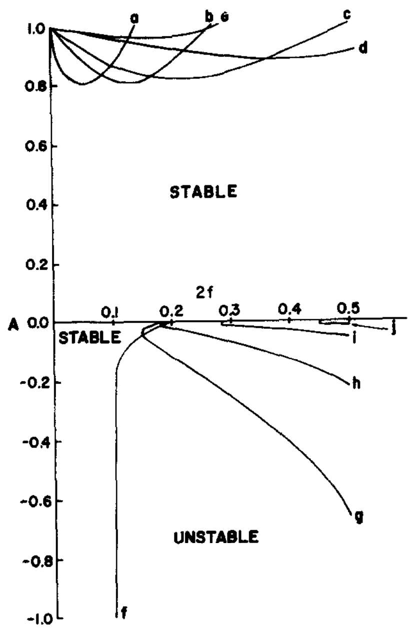


FIG. 4. Domain of stability of scheme (C.12). Stable for $|A| \le 1, f \le \frac{1}{2}, 2\alpha = 0$. For $A > 0$, stable below lines for: $a$, $2\alpha = \frac{1}{18}\pi$; $b$, $2\alpha = \frac{1}{9}\pi$; $c$, $2\alpha = \frac{8}{9}\pi$; $d$, $2\alpha = \frac{1}{2}\pi$; $e$, $2\alpha = \frac{11}{12}\pi$. For $A < 0$, stable left of lines for: $f$, $2\alpha = \frac{1}{36}\pi$; $g$, $2\alpha = \frac{1}{6}\pi$; $h$, $2\alpha = \frac{5}{18}\pi$; $i$, $2\alpha = \frac{1}{2}\pi$; $j$, $2\alpha = \frac{7}{9}\pi$. Unstable area decreases as $2\alpha \to \pi$.

## D. *Advection-Diffusion-Inertial Equation*

Combining previously examined stable schemes to approximate

$$\frac{\partial w}{\partial t} + \hat{A}\frac{\partial w}{\partial x} = \hat{K}\frac{\partial^2 w}{\partial x^2} - i\hat{f}w$$

results in the following 2 two-step schemes. Both schemes are first-order in time and second-order in space. Their properties are listed below.

$$W_j^{n+1} = \overline{W_j^n} - A\delta_2 W_j^n + K\delta_1^2 W_j^n - ifW_j^{n+1} \tag{D.1}$$

$$W_j^{n+2} = W_j^n - 2A\delta_2 W_j^{n+1} + 2K\delta_1^2 W_j^n - 2ifW_j^{n+2}$$

$$e = O(\Delta t) + O[(\Delta x)^2]$$

$$G(1 + 4f)(1 + f) = (1 + 4f) - 3fA\sin 2\alpha + (1 - 2f)A^2(\cos 2\alpha - 1)$$
$$- 6fAK(\sin 2\alpha - 2\sin\alpha) + (4 + 16f)K(\cos\alpha - 1) - i[2f + 8f^2$$
$$+ (1 - 2f^2)A\sin 2\alpha + 3fA^2(\cos 2\alpha - 1)$$
$$+ (2 - 4f^2)AK(\sin 2\alpha - 2\sin\alpha) + (8f + 32f^2)K(\cos\alpha - 1)].$$

Stability conditions are not easily found for either (D.1) or (D.2). The magnitudes of $G$ for various values of $A$, $K$, and $f$ are obtained numerically. The domain of stability of equation (D.1) is given in Fig. 5 for $\hat{f}\Delta t$ equal to .001, .006, .06, and .18. When $K = 0$, (D.1) reduces to (C.11).

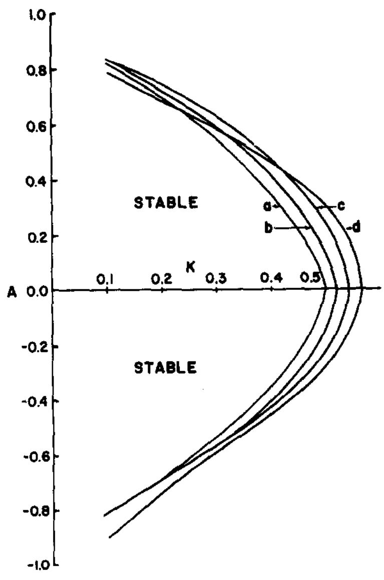


FIG. 5. Domain of stability of scheme (D.1). Stable left of lines for: $a, \hat{f}\Delta t = .001$; $b, \hat{f}\Delta t = .006$; $c, \hat{f}\Delta t = .06$; $d, \hat{f}\Delta t = .18$. For $K = 0$, (D.1) reduces to (C.11) (Fig. 3).

$$W_j^{n+1} = \overline{W_j^n} - A\delta_2 W_j^n + K\delta_1^2 W_j^n - \tfrac{1}{2}\, if\,(W_j^{n+1} + \overline{W_j^n}) \qquad \text{(D.2)}$$

$$W_j^{n+2} = W_j^n - 2A\delta_2 W_j^{n+1} + 2K\delta_1^2 W_j^n - if\,(W_j^{n+2} + W_j^n)$$

$$e = O(\Delta t) + O[(\Delta x)^2]$$

$$\begin{aligned} G(1+f^2)(1+\tfrac{1}{4}f^2) &= 1 - \tfrac{3}{4}f^2 - \tfrac{1}{4}f^4 + (\tfrac{1}{2}f^3 - \tfrac{5}{2}f)A\sin 2\alpha \\ &+ (1-\tfrac{1}{2}f^2)A^2(\cos 2\alpha - 1) - 3fAK(\sin 2\alpha - 2\sin\alpha) \\ &+ (4+f^2)(\cos\alpha - 1) - i[2f + \tfrac{1}{2}f^3 + (1-2f^2)A\sin 2\alpha \\ &+ \tfrac{3}{2}fA^2(\cos 2\alpha - 1) + (2-f^2)AK(\sin 2\alpha - 2\sin\alpha) \\ &+ (4f+f^3)(\cos\alpha - 1)]. \end{aligned}$$

Numerical calculations of $G$ show that scheme (D.2) is unstable for short wavelengths (approximately less than $6\Delta x$). The values of $\hat{f}\,\Delta t$. checked are .006, .06, .18, and .36. When $f = 10^{-4}$ sec$^{-1}$ these values correspond to 1-minute, 10-minute, 30-minute, and 1-hour time steps, respectively.

## IV. CONCLUSIONS

Twenty-two difference schemes are examined for von Neumann necessary condition for stability. All are second-order in space. Both first- and second-order time approximations are investigated.

Of the five approximations to the diffusion equation investigated, two are found to be stable for the same values of $K$ as the first-order forward difference scheme and one is found to be stable under a more restrinctive condition. The remaining two are found to be unstable for short wavelengths. One second-order and one first-order scheme approximating the advection-diffusion equation are shown to be stable while the other scheme examined is shown to be unstable. Eight one-step and four two-step schemes approximating the advection-inertial equation are studied. Of these two are found to be slightly unstable with very small amplification after 10000 iterations. Six first-order and two second-order, both of leap frog type, schemes are found to be stable. For the advection-diffusion-inertial equation, 2 two-step schemes are formed from the preceding stable schemes. The analysis shows that such combinations are not necessarily stable.

In general, the difference equations examined are to be applied to nonlinear partial differential equations. The stability criteria presented here are based on the linear equations with constant coefficients, and may therefore differ from the stability properties of the nonlinear equations. It is hoped that some actual computations applying these schemes to the nonlinear problems can be performed in the future.

When stable difference schemes for individual advection, diffusion, or inertial terms are combined to approximate partial differential equations containing several of these terms, there seems to be no simple method to determine the stability of the resulting equation without actually calculating the eigenvalues of its amplification matrix. Similarly, the stability of a two-step scheme does not seem to be related in any obvious way to the stability of each step. In Section III.C, 2 two-step schemes are presented each of which consists of two stable one-step schemes. However, when these two-step schemes are examined for stability, one is found to be stable and the other unstable. These examples support Kasahara's [3] remark that one should examine the stability of a complete scheme rather than guess a stability condition inclusive of all stability conditions of various simplifications of the complete equation.


## Acknowledgments

The author is indebted to Drs. A. Kasahara, W. Washington, and J. Gary for their advice, comments, and explanations during the course of this work; also to Drs. A. Kasahara and E. Lorenz for reading the manuscript carefully.


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